Today we will learn what is thermal radiation

Laws of thermodynamics

Thermal Equilibrium

Radiative Diffusion Equation

Blackbody and Thermal Radiation

Useful to describe radiation emitted by stars, accretion disks, nebulae, stellar atmospheres, etc...
Laws of Thermodynamics

**First Law**: Energy can be changed from one form to another, but it cannot be created or destroyed. (Conservation of Energy)

\[ \Delta U = Q - W \]

- Change in internal energy
- Heat added to the system
- Work done by the system
Laws of Thermodynamics

**Second Law**: Heat can never pass from a colder to a warmer body without some other change, connected therewith, occurring at the same time (Clausius formulation)

\[ \Delta S = \text{Entropy} = \frac{\Delta Q}{T} \]

\[ \Delta Q \quad \text{Heat Transfer} \]

\( T_1 \) (hot) \quad \text{to} \quad \text{(cold)} \quad T_2 \]
Thermal Equilibrium

Thermal Radiation is radiation emitted by matter in thermal equilibrium.

Thermal Equilibrium: Two physical systems are in thermal equilibrium if no heat flows between them when they are connected by a path permeable to heat.

Important: there is a reason why I highlighted the word matter. Indeed radiation does not necessarily need to be in thermal equilibrium to have thermal radiation.
Matter in Thermal Equilibrium

Suppose to have a plasma in thermal equilibrium (thermal plasma). What does this mean in terms of micro-physical properties of the matter?

Probability distribution function of (non-relativistic) velocities is the Maxwell-Boltzmann distribution:

\[
F(v) \, dv = 4\pi v^2 \left(\frac{m}{2\pi k T}\right)^{3/2} e^{-mv^2/(2kT)} \, dv
\]
Matter in Thermal Equilibrium

The Maxwell-Boltzmann distribution written in this way is valid only for non-relativistic particles. We will find many astrophysical systems where particles have relativistic speeds and they emit thermal radiation.

\[ p = \gamma \beta mc \]

\[ F(p) \, dp = \frac{p^2 e^{-\gamma \Theta}}{\Theta m^3 c^3 K_2(1/\Theta)} \, dp \]

This form is valid in BOTH the relativistic and non-rel. limit.

Modified Bessell Function of the 2nd kind
Blackbody Radiation

1. Take an enclosure (of arbitrary shape) at temperature $T$ and do not let radiation flow in or out until equilibrium is achieved. Matter and radiation are in this case in thermal equilibrium.
2. Now open a small hole in the first enclosure so that you do not disturb equilibrium.
3. Join a second enclosure with the same properties as the first (temperature $T$, arbitrary shape) and place a “filter” between the holes so that only a specific frequency can pass through it.

What will happen?
We have said nothing about the emissivity and absorption properties of the cavity walls. Why?

Should we expect the radiation field inside the cavity to depend on the properties of the wall or not?
Blackbody Radiation

If radiation flows between the two systems you've just violated the second law of thermodynamics since the two enclosures have the same temperature! So no *net* radiation will flow between the enclosures

This is true regardless of the properties of the cavity!
The conclusion is therefore simple: the specific brightness of your enclosure must be a universal function of temperature and frequency alone.

\[ I_\nu = B_\nu(T) \]

This universal function is called the Planck function.
Properties of the Planck Spectrum

\[ B_\nu(T) = \frac{2h \nu^3 / c^2}{\exp(h \nu / kT) - 1} \]

We will not derive this function here, but read the Section “The Planck Spectrum” on the textbook R&L.
Properties of the Planck Spectrum

\[ B_\nu(T) = \frac{2\hbar \nu^3/c^2}{\exp(h\nu/kT) - 1} \]

The “2” is there because light has two polarizations (left and right circular polarization)
Properties of the Planck Spectrum

\[ B_\nu(T) = \frac{2\hbar \nu^3/c^2}{\exp(h\nu/kT) - 1} \]

“\( h \) of course is telling you that photons are quantized"
Properties of the Planck Spectrum

\[ B_\nu(T) = \frac{2h \nu^3 c^2}{\exp\left(\frac{h \nu}{kT}\right) - 1} \]

“c” is there because of the propagation velocity of photons
Properties of the Planck Spectrum

\[ B_\nu(T) = \frac{2h \nu^3/c^2}{\exp(h \nu/kT) - 1} \]

This term comes from the fact that photons are **bosons** *(integer spin particles)* so they obey the Bose-Einstein statistics.

\( h\nu \) is the energy of a photon

The chemical potential of photons is zero (otherwise you'd have seen there \((h\nu - \mu)/kT\))
Properties of the Planck Spectrum

\[ B_\nu(T) = \frac{2\hbar \nu^3 / c^2}{\exp \left( \frac{\hbar \nu}{kT} \right) - 1} \]

The “-1” is there because photons are bosons, so there can be multiple photons with the same quantum number.
Properties of the Planck Spectrum

\[ B_\nu(T) = \frac{2\hbar \nu^3 / c^2}{\exp(h\nu/kT) - 1} \]

The frequency appears there in this way because it can be easily demonstrated that the density of states is proportional to \( 2\nu^2/c^3 \). You then need to multiply this by the average energy per state:

\[ E_{\text{avg}} = \frac{h\nu}{\exp(h\nu/kT) - 1} \]
Kirchoff's Law

If you place a body with temperature $T$ inside the cavity, what will happen?

The cavity + body is still a blackbody enclosure at temperature $T$ (remember we just said that it does not matter how the enclosure is made)

Therefore the source function of the body is:

$$S_\nu = B_\nu(T) \rightarrow j_\nu = \alpha_\nu B_\nu(T)$$

**Thermal Radiation**

**Blackbody Radiation**
As we have stated before an example of thermal radiation is blackbody radiation. But the opposite is not generally true: thermal radiation is not necessarily blackbody radiation.

Blackbody $\rightarrow$ Thermal

Thermal $\not\rightarrow$ Blackbody

*Blackbody radiation is generated by an optically thick medium emitting thermal radiation*
Understanding Kirchhoff’s Law

Now we are in position to understand Kirchhoff’s laws in the light of the equation of radiative transfer.

1. A hot dense gas produces light with a continuous spectrum.
2. A hot dense gas surrounded by a cool tenuous gas produces light with a continuous spectrum which has gaps at discrete wavelengths.
3. A hot tenuous gas produces light with emission lines at discrete wavelengths.
Understanding Kirchhoff’s Law

1. A hot dense gas produces light with a continuous spectrum.

Consider (for simplicity) that the initial surface brightness is zero \( I_\nu(0) = 0 \)

Then: \( I_\nu = S_\nu (1 - e^{-\tau_\nu}) \)

Since the gas is dense, the optical depth is >>1. Therefore: \( I_\nu = S_\nu = B_\nu \)

And this implies that what you will see is a blackbody spectrum (Planck spectrum)
Understanding Kirchhoff’s Law

2. A hot dense gas surrounded by a cool tenuous gas produces light with a continuous spectrum which has gaps at discrete wavelengths.

In this case there is a background (initial) specific emissivity, i.e., the one of the hot gas which we know is a blackbody. Therefore: \( I_\nu(0) = B_\nu \)

The cold gas instead is not emitting (or, more correctly, has negligible emission when compared to the hot gas and thus: \( S_\nu \approx 0 \)

Therefore: \( I_\nu = B_\nu e^{-\tau_\nu} \)

The intensity is a Planck continuum, lowered where the optical depth is high. And where does the optical depth become high? Only at those frequencies that correspond to the absorption energies of the cold atoms. Thus, we see absorption lines.
Understanding Kirchhoff’s Law

3. A hot tenuous gas produces light with emission lines at discrete wavelengths

In the optically thin case \( I_v = S_v (1 - e^{-\tau_v}) \approx S_v (1 - 1 + \tau_v) = S_v \tau_v \)

The intensity will be high where the optical depth is high. Since there is no background intensity, these are seen as emission lines.
Principle of Detailed Balance

At equilibrium, each elementary process should be equilibrated by its reverse process.

This principle has been derived here by using Kirchhoff’s law, which requires that matter is in thermodynamic equilibrium. However, as we will see now, it is valid also for nonthermal emission.
The Einstein Coefficients

Since Kirchhoff’s law relates emission and absorption of a body, there must be some microscopic relation between these two mechanisms. Einstein found out which and how these mechanisms operate. The Einstein coefficients are important because they generalize Kirchhoff’s law to matter that is not in thermodynamic equilibrium.

Let’s go back to what we said last week:

Transition probabilities

- $A_{21} \ [s^{-1}] = \text{transition probability for spontaneous emission per unit time}$
- $B_{12} J_\nu = \text{transition probability for absorption per unit time}$
- $B_{21} J_\nu = \text{transition probability for stimulated emission per unit time}$
- The last two depend on the strength of the radiation field: $J_\nu = (1/4\pi) \int I_\nu d\Omega$.  

Einstein coefficients
The Einstein Coefficients

\[
n_1 B_{12} \bar{J} = n_2 A_{21} + n_2 B_{21} \bar{J}.
\]

\[
\bar{J} \equiv \int_0^\infty J_\nu \phi(\nu) d\nu.
\]

\[
\bar{J} = \frac{A_{21}/B_{21}}{(n_1/n_2)(B_{12}/B_{21}) - 1}
\]

(The function \(\phi(\nu)\) is the so-called line profile. In simple words, if you think about a line, either in absorption or emission, it has always a certain (narrow) width. The profile can be described by different functions (e.g., Gaussian, Lorentzian) which carry information on the microphysics of the atoms. For example, an absorption line with a Gaussian line profile is generated by collisions of atoms that absorb radiation at a frequency \(\nu_0\) and small neighbor frequencies due to Doppler effects. We’ll see this more carefully in the Lecture 11).
The Einstein Coefficients

We know that in thermodynamic equilibrium the ratio between $n_1$ and $n_2$ is given by the Boltzmann relations:

$$\frac{n_1}{n_2} = \frac{g_1 \exp\left(-\frac{E}{kT}\right)}{g_2 \exp\left[-\left(E + h\nu_0\right)/kT\right]} = \frac{g_1}{g_2} \exp\left(h\nu_0/kT\right)$$

But we also know that in thermodynamic equilibrium $J_\nu = B_\nu$ and therefore:

$$\bar{J} = \frac{A_{21}/B_{21}}{\left(g_1 B_{12}/g_2 B_{21}\right) \exp(h\nu_0/kT) - 1}$$

We can say that $\bar{J} = B_\nu$ because the width of the line is small and the intensity is varying very slowly across this width. Therefore:

$$g_1 B_{12} = g_2 B_{21},$$

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21}.$$
The Einstein Coefficients

The properties just derived, i.e. \[
\begin{aligned}
g_1 B_{12} &= g_2 B_{21}, \\
A_{21} &= \frac{2hv^3}{c^2} B_{21}.
\end{aligned}
\]

are independent on the existence of thermal equilibrium. Therefore they hold in any circumstance because they refer only to atomic properties of the emitting matter and we have made no reference to any temperature throughout the derivation. These Einstein relations are an extension of Kirchhoff’s law. The R&L shows what happens when nonthermal radiation is involved.

QUESTION: You might wonder why I say we never mentioned the temperature and we never required the existence of thermal/thermodynamic equilibrium whereas we just said in the previous slide that we assume thermodynamic equilibrium between the atoms and so we use the Boltzmann relations. Can you find out why?
Thermodynamics of Blackbody Radiation

We can derive an important property of blackbody radiation by treating it as a photon gas and then applying the laws of thermodynamics.

\[
dQ = dU + pdV \quad \text{1st law of thermodynamics}
\]

\[
dS = \frac{dQ}{T} \quad \text{2nd law of thermodynamics}
\]

We have seen in the first lecture how to calculate \( u \), the energy density of radiation:

\[
u = \frac{4\pi}{c} \int J_\nu d\nu \quad \text{And here} \quad J_\nu = B_\nu(T)
\]

We thus know what the total energy of radiation is:

\[
U = uV
\]

We have also seen how to calculate the radiation pressure \( p \), which is 1/3 the energy density:

\[
p = \frac{u}{3}
\]
Thermodynamics of Blackbody Radiation

\[
dS = \frac{V}{T} \frac{du}{dT} dT + \frac{u}{T} dV + \frac{1}{3} \frac{u}{T} dV = \frac{V}{T} \frac{du}{dT} dT + \frac{4u}{3T} dV
\]

From this expression you see that \( dS \) is an exact differential. Let’s remember the definition of exact differential. A differential \( df \) is exact (or perfect) if:

\[
df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.
\]

Therefore:

\[
\left( \frac{\partial S}{\partial T} \right)_V = \frac{V}{T} \frac{du}{dT}
\]

\[
\left( \frac{\partial S}{\partial V} \right)_T = \frac{4u}{3T}
\]

Let’s now differentiate the first partial derivative again with respect to \( V \):

\[
\frac{\partial^2 S}{\partial T \partial V} = \frac{1}{T} \frac{du}{dT} = -\frac{4u}{3T^2} + \frac{4}{3T} \frac{du}{dT}
\]

\[
\frac{du}{u} = 4 \frac{dT}{T}
\]
Thermodynamics of Blackbody Radiation

Solution: \[ \log u = 4 \log T + \log a, \quad \rightarrow \quad u(T) = aT^4 \]

This is the Stefan-Boltzmann law.
For isotropic radiation:

\[ u = \frac{4\pi}{c} \int B_\nu(T) \, d\nu = \frac{4\pi}{c} B(T) = \frac{ac}{4\pi} T^4 \]

But there is another result which is very profound and comes from this discussion: blackbody radiation is the type of radiation with maximum possible entropy.

What is the entropy of blackbody radiation? Since \( u(T) = aT^4 \) we can write:

\[ \left( \frac{\partial S}{\partial T} \right)_\nu = 4aVT^2, \quad \left( \frac{\partial S}{\partial V} \right)_T = \frac{4}{3} aT^3 \]

We can thus write: \[ S = \left( \frac{4}{3} \right) aVT^3 + \text{constant} \]

(here the constant is actually zero, since \( S \to 0 \) when \( T \to 0 \)).
Stellar Spectra: blackbody emitters

The Sun

extraterrestrial solar spectral irradiance
total area: 1367 W/m²

blackbody spectrum for T = 5777 K
total area: 1367 W/m²
Cosmic Microwave Background

Deviation from a 2.725 K Blackbody: 1/100,000
Neutron Stars

1E1048.1—5937

6-s period Anomalous X-ray Pulsar

Check also the Magnificent Seven:

http://en.wikipedia.org/wiki/The_Magnificent_Seven_(neutron_stars)
Accretion Disks

Used to infer the Black Hole spin period (and test General Relativity)

McClintock, Narayan & Steiner (2013)
Accretion Disks
Accretion Disks
If particles and radiation are in equilibrium it means that there is a steady-state condition: there is no net flow of energy in nor out of a volume element, nor any transfer of energy between matter and radiation.

Every process, such as the absorption of a photon, occurs at the same rate as its inverse process, such as the emission of a photon. Such an idealised condition is referred to as thermodynamic equilibrium. A blackbody is by definition in thermodynamic equilibrium.

You might now wonder how is possible that a star or an accretion disk, or a neutron star emits like a blackbody. For example in a star there is a net outward flow of energy. The temperature varies from millions of degrees in the core to thousands of degrees in the atmosphere. So we do we see blackbody-like emission?
Suppose that the typical distance traveled by particles and photons between collisions—their mean free path—is small compared to the scale over which the temperature changes significantly. This situation applies to most of the stellar interior, where density and temperature are high, so that the mean distance between collisions is small. You can think of this situation as the particles and photons being confined to a limited volume of nearly constant temperature.

In such cases it is possible to derive a simple expression for the energy flux, relating it to the local temperature gradient. This result is called the Rosseland approximation.
### Summary of Radiation Properties

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<th>Thermal</th>
<th>Blackbody</th>
<th>Bremsstrahlung</th>
<th>Synchrotron</th>
<th>Inverse Compton</th>
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<tbody>
<tr>
<td><strong>Optically thick</strong></td>
<td>–</td>
<td>YES</td>
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<tr>
<td><strong>Maxwellian</strong></td>
<td>YES</td>
<td>YES</td>
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<tr>
<td><strong>distribution</strong></td>
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<td><strong>of velocities</strong></td>
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<td><strong>Relativistic</strong></td>
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<tr>
<td><strong>speeds</strong></td>
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<tr>
<td><strong>Main Properties</strong></td>
<td>Matter in thermal equilibrium</td>
<td>Matter AND radiation in thermal equilibrium</td>
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</tbody>
</table>

**Rule of thumb:** Blackbody is always thermal, but thermal radiation is not always blackbody

Blackbody → **thermal** radiation
## What We Have Seen So Far

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<th>Topic</th>
<th>Description</th>
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<td>Some definitions (specific brightness, flux, energy density, radiation pressure, etc...).</td>
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<tr>
<td>Constancy of Specific Intensity in Free Space</td>
<td>Brightness does not depend on distance (in free space)</td>
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<td>The Radiative Transfer Equation</td>
<td>How specific intensity varies with absorption, emission and scattering. Optical depth. Some simple analytical solution</td>
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<tr>
<td>Thermal Radiation: blackbody radiation and Kirchhoff’s law</td>
<td>Thermal radiation emerges when the matter is in thermal equilibrium. Blackbody radiation emerges when matter AND radiation are in thermal equilibrium.</td>
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<tr>
<td>Stefan-Boltzmann law and entropy of blackbody radiation</td>
<td>Blackbody is the maximum entropy radiation. Stefan-Boltzmann law can be derived from simple thermodynamics</td>
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<td>Properties of Blackbody Radiation</td>
<td>Wien law, Rayleigh-Jeans law, perfect emitter</td>
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<tr>
<td>Einstein Coefficients</td>
<td>Relation between Einstein coefficients is an extension of Kirchhoff's law for systems which are not necessarily in thermal equilibrium</td>
</tr>
<tr>
<td>Local Thermodynamic Equilibrium and Rosseland Mean Opacity</td>
<td>Real bodies are never blackbodies, but many systems are very good approximations since there is LTE.</td>
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